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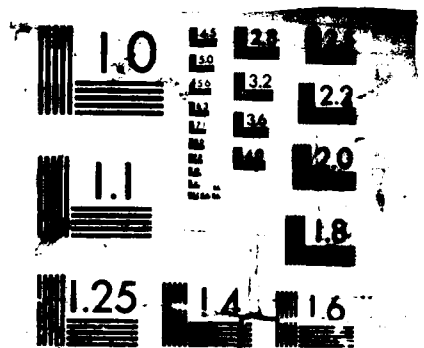
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NONLINEAR PARAMETRIC LEAST-SQUARES ADJUSTMENT

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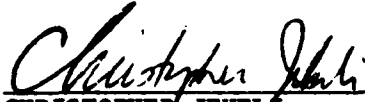
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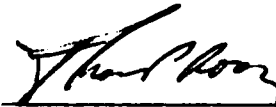
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A parametric adjustment model expresses n observables in terms of u parameters, where the structure linking the two groups is in general nonlinear. This model can be expanded in the Taylor series based on an "initial point" P. The standard procedure includes only the linear terms, constituting a known linearized model at P which is resolved upon applying the least-squares (L.S.) criterion. If necessary, the solution is iterated, i.e., the initial point in a given step is revised using the L.S. result from the previous step.		

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## 1. INTRODUCTION

The basic mathematical model of the parametric adjustment expresses each of the observables in terms of parameters, where the structure tying together the two groups of variables is in general nonlinear. The present study addresses the resolution of such an adjustment problem through an isomorphic geometrical setup with tensor structure and notations. The number of observables is denoted by  $n$  and the number of parameters by  $u$ , where  $n > u$  for an adjustment to take place. The indices identifying the elements in either group are written as superscripts, conforming to tensor notations for coordinates. Thus, the observables in the parametric model are represented by

$$x^r = x^r(u^\alpha) , \quad (1)$$

where the Roman-letter and the Greek-letter indices vary respectively as

$$r = 1, 2, \dots, n ; \quad \alpha = 1, 2, \dots, u . \quad (2a,b)$$

The actual observations of  $x^r(u^\alpha)$  are in general inconsistent with the parameters  $u^\alpha$ , giving rise to the discrepancies

$$dx^r = x^r - x^r(u^\alpha) . \quad (3)$$

Here and in the sequel, the symbol  $x^r$  is reserved for the observations, while the observables are represented by  $x^r(u^\alpha)$  or other symbols distinct from  $x^r$ .

In a standard procedure, (1) is subject to the Taylor expansion

$$x^r(u^\alpha) = x_o^r + [\partial x^r / \partial u^\alpha]_o du^\alpha + (1/2)[\partial^2 x^r / \partial u^\alpha \partial u^\beta]_o du^\alpha du^\beta + \dots , \quad (4a)$$

$$du^\alpha = u^\alpha - u_o^\alpha , \quad (4b)$$

where the tensor symbolism implies the summation convention over the dummy (repeating) indices. The subscript "o" corresponds here to the initial values of the parameters,  $u_o^\alpha$ , so that  $x_o^r = x^r(u_o^\alpha)$ . Upon introducing the notation  $dx^r$ , such that

$$dx^r = x^r - x_o^r , \quad (5)$$

(3) in conjunction with (4a) form the nonlinear observation equations

$$dx^r = [\partial x^r / \partial u^\alpha]_o du^\alpha + (1/2)[\partial^2 x^r / \partial u^\alpha \partial u^\beta]_o du^\alpha du^\beta + \dots + dx^r . \quad (6)$$

The Gauss' form of a two-dimensional surface ( $u=2$ ) embedded in a three-dimensional flat space ( $n=3$ ) is described, together with two other forms, in Chapter 6 of [Hotine, 1969]. In [Blaha, 1984], which treated only the linearized parametric adjustment, both the  $n$ -dimensional "observational" space and the  $u$ -dimensional "model" surface were considered flat. The latter was thus in reality a hyperplane. Although the model surface is now intrinsically a curved space, the surrounding observational space can again be considered flat. This stems from the representation of variance-covariance matrices by associated metric tensors and of weight matrices by metric tensors as demonstrated in [Blaha, 1984]. Since the variance-covariance and the weight matrices do not change with the coordinates  $u_0^\alpha$  as discussed in the preceding paragraph, the associated metric tensor  $g^{rs}$  and the metric tensor  $g_{sr}$  for the observational space are constant, at least insofar as displacements along the model surface are concerned. And in the absence of any conflicting information or restriction, these tensors are assumed constant throughout the observational space. The latter is thus flat as an admissible and tangible outcome.

In general, a flat space can be described via Cartesian coordinates. In a Cartesian coordinate system, a given point is depicted by a set of coordinates  $x^r$ ,  $r=1,2,\dots,n$ , which can be interpreted as its position vector ( $\rho$ ) expressed by contravariant components,  $\rho^r = x^r$ . Considered as a tensor, and thus as a point function,  $\rho^r$  is associated with this given point (and not, for example, with the origin). Accordingly, if the observational-space coordinates were Cartesian,  $x_0^r = x^r(u_0^\alpha)$  would represent the position vector associated with the model-surface point called P. Similarly, (1) would express a family of position vectors associated with the model surface. Any of these position vectors could be freely parallel-transported to the point P and eventually give rise to tensor equations there. Although derived in Cartesian coordinates, such equations would be valid in any coordinates applicable to a flat space.

In multiplying the Cartesian coordinates  $x^r = x^r(u^\alpha)$  by the appropriate transformation factors, one obtains the contravariant components of the same family of position vectors in another coordinate system. Equation (1) can thus be broadly interpreted as expressing a family of position vectors, whose functional relationship to the coordinates  $u^\alpha$  depends in part on the above transformation factors, i.e., on the observational-space coordinate system in use. If the latter were completely general, the position vectors could not be



## 2. GEOMETRICAL RESOLUTION

### 2.1 Geometrical Setup

The geometrical setup corresponding to the nonlinear adjustment model is schematically illustrated in Fig. 1. In accordance with the discussion in the Introduction, the observations are interpreted as the contravariant components of the position vector  $x^r$  of the point denoted  $Q$ . The observables consistent with the adjustment model correspond to a family of position vectors  $x^r(u^\alpha)$  to the model surface. In particular,  $x_0^r = x^r(u_0^\alpha)$  identifies the position vector to the point  $P$ , and  $\bar{x}^r = x^r(u^\alpha(\bar{P}))$  identifies the position vector to the point  $\bar{P}$ , where the model-surface coordinates of  $P$  and  $\bar{P}$  are  $u_0^\alpha$  and  $u^\alpha(\bar{P})$ , respectively. The difference in the model-surface coordinates between  $\bar{P}$  and  $P$  is denoted by  $du^\alpha$ , so that

$$du^\alpha = u^\alpha(\bar{P}) - u_0^\alpha. \quad (8)$$

In the course of the development, the model-surface point  $\bar{P}$  is considered variable, and is restricted to a small neighborhood of the fixed point  $P$ . Accordingly, the coordinate differences  $du^\alpha$  are variable. As two special cases,  $\bar{P}$  can coincide with  $P$ , or can coincide with the point denoted  $\bar{P}$  in Fig. 1.

The differences in the observational-space coordinates between  $\bar{P}$  and  $P$  are denoted as

$$\Delta x^r = \bar{x}^r - x_0^r, \quad (9)$$

which also describes the differences in the contravariant components of the position vectors belonging to two distinct points in space, namely  $\bar{P}$  and  $P$ . In general, unless these position vectors are parallel-transported to a common point, (9) is not a tensor equation and  $\Delta x^r$  is not a tensor. However, in the special observational-space coordinates with a constant metric tensor, either of these vectors can be freely parallel-transported to any location. It is then convenient to regard  $\bar{x}^r$  as parallel-transported from  $\bar{P}$  to  $P$ , in which case (9) is a tensor equation at  $P$  and  $\Delta x^r$  is a tensor at  $P$ . This will allow  $\Delta x^r$  to participate in other tensor equations at  $P$ . The geometrical object described by the contravariant components  $\Delta x^r$  can be symbolized by an arrow from  $P$  to  $\bar{P}$ , referred to as the vector  $P\bar{P}$ . In Fig. 1, this vector is denoted as  $\Delta x$ , and its tip depicts one special location of the point  $\bar{P}$ .

The differences in the observational-space coordinates between Q and P are denoted by

$$dx^r = x^r - x_0^r, \quad (10)$$

which has already appeared as equation (5). In analogy to the above,  $dx^r$  also represents the differences in the contravariant components of the position vectors belonging to Q and P, respectively. Since the position vector belonging to Q can again be freely parallel-transported to P, (10) is a tensor equation at P and  $dx^r$  is a tensor at P. The geometrical object described by  $dx^r$  is symbolized in Fig. 1 by the vector  $dx$ . Similarly, the differences in the observational-space coordinates between Q and  $\bar{P}$  are denoted by

$$d\bar{x}^r = x^r - \bar{x}^r, \quad (11)$$

which is essentially equation (3). The position vector belonging to Q can be freely parallel-transported to  $\bar{P}$ , in which case (11) is a tensor equation at  $\bar{P}$  and  $d\bar{x}^r$  is a tensor at  $\bar{P}$ . The geometrical object described by  $d\bar{x}^r$  is symbolized in Fig. 1 by the vector  $d\bar{x}$ . As another possibility, both position vectors can be freely parallel-transported to P, in which case (11) is a tensor equation at P and  $d\bar{x}^r$  is a tensor at P. In terms of Fig. 1, the geometrical object described by this tensor can be imagined as a vector of the same magnitude and orientation as the vector  $d\bar{x}$  already depicted, but drawn from P.

The latter possibility is particularly useful. It results in three tensor equations at P, namely (9), (10), and (11), yielding

$$dx^r = \Delta x^r + d\bar{x}^r, \quad (12)$$

which is again a tensor equation at P. Since the numerical values  $x^r$  are given (they represent the actual observations), and the numerical values of  $x_0^r$  are fixed as a function of the fixed coordinates  $u_0^\alpha$ , the tensor  $dx^r$  is known from (10) as a part of the geometrical setup. However, the numerical values of  $\bar{x}^r$  are unknown (they represent the adjusted observations), and so are the values of  $\Delta x^r$  and  $d\bar{x}^r$ . On the other hand,  $\Delta x^r$  can be expressed as a function of the coordinate differences  $du^\alpha$  in (8). Since these differences are linked to the position of the variable point  $\bar{P}$ , and since the tensor  $d\bar{x}^r$  is also linked to the position of  $\bar{P}$ , it follows that a variety of realistic criteria pertaining to  $d\bar{x}^r$  could lead to the determination of  $du^\alpha$  and thus to the resolution of the entire setup.

this choice appealing geometrically for its simplicity and objectivity, but it represents at the same time the desired least-squares (L.S.) solution. Since the residual vector  $V$  from adjustment calculus corresponds to  $-dx^r$ , and the adjustment weight matrix  $P$  corresponds to  $g_{sr}$ , the familiar L.S. quadratic form,  $V^T P V = \text{minimum}$ , corresponds to

$$dx^s g_{sr} dx^r = ds^2 = \text{minimum} ,$$

where  $ds$  is the length of the vector  $dx$ . But the minimum-length criterion for  $dx$  means that this vector must be orthogonal to the model surface. In accordance with most practical cases, we assume that the small neighborhood of  $P$  contains only one model-surface point satisfying the orthogonality requirement, and that the latter entails a minimum, rather than a maximum, length  $ds$ . A situation of this kind is reflected in the top portion of Fig. 1, indicating that one is concerned with the point  $\bar{P}$  and not with  $Z$ .

## 2.2 Design Tensor

We begin with the system of equations (1), interpreted in the Introduction as the Gauss' form of a  $u$ -dimensional surface embedded in an  $n$ -dimensional flat space. This surface has been called the model surface, and the surrounding space has been called the observational space. With respect to a given surface point  $P$ , an infinitesimal displacement along the surface can be described either in the surface coordinates by the coordinate differences  $du^\alpha$ ,  $\alpha=1,2,\dots,u$ , or in the space coordinates by the coordinate differences  $dx'^r$ ,  $r=1,2,\dots,n$ . The two are related by the ordinary formula for total differentiation,

$$dx'^r = (\partial x'^r / \partial u^\alpha) du^\alpha , \quad (17)$$

where the partial derivatives are evaluated at  $P$ .

Although the surface coordinate system is in general curvilinear,  $du^\alpha$  can be regarded not only as a set of coordinate differences, but also as contravariant components of a small vector. Such a dual role is explained by the statement on page 5 of [Hotine, 1969]: "Over short distances, we can, nevertheless, consider that the coordinate lines are straight in a curvilinear system. By analogy with the Cartesian definition, we still can say that a small change in coordinates...represents the...contravariant components of a small

where  $\delta_{\beta}^{\alpha}$  is the Kronecker delta, we can write

$$A_{\beta}^r \equiv \partial x^r / \partial u^{\beta} = \ell^r \ell_{\beta} + j^r j_{\beta} + \dots,$$

and thus confirm (14a).

In retracing the above derivation, we conclude that the dual role of  $du^{\alpha}$  (and also of  $dx'^r$ ) as a set of small coordinate differences and as a tensor at the surface point P has likewise resulted in a dual role of  $A_{\alpha}^r$ . Specifically, as  $\partial x^r / \partial u^{\alpha}$  it can be regarded as a set of partial derivatives relating the space and the surface coordinates at P. And as  $\ell^r \ell_{\alpha} + j^r j_{\alpha} + \dots$  it can be regarded as a mixed tensor at P, transforming like a space tensor in the contravariant indices and like a surface tensor in the covariant indices. Except for the higher-dimensionality of (14a), this equation has the same form as the tensor equation 6.09 in [Hotine, 1969], although the latter was derived by other means. The important property of (14a) in tensor form is that it is based on the space and the surface components of the orthonormal vectors  $\ell, j, \dots$  at P, which, in geometrical interpretation, span the model plane. These vectors are quite arbitrary, in the sense that any such set leads to the same tensor  $A_{\alpha}^r$ . Due to its correspondence to the design matrix A in adjustment calculus, which can be noticed upon comparing (7) and (7'),  $A_{\alpha}^r$  is called the design tensor.

### 2.3 Least-Squares Solution

Since the values of the coordinate differences  $du^{\alpha}$  are small by construction, a good approximation to the final solution  $du^{\alpha}$  can be obtained through a linearized model at P. In the geometrical context, such a model corresponds to (16) without the derivatives beyond the first order, i.e., to

$$dx^r = A_{\alpha}^r du^{\alpha} + dx''^r, \quad (19)$$

where the notation  $dx^r$  has been replaced by  $dx''^r$ . In terms of Fig. 1, the vector  $dx$  remains unaltered, but the model surface is replaced by the model plane. If the adjustment model were indeed linear,  $A_{\alpha}^r$  would be constant throughout the model plane due to  $\Omega_{\alpha\beta}^r \equiv 0$  in (14b). The model surface metric tensor at P would be formed as

Upon comparing (21) and (22), we assert that the L.S. solution (22) of the linear model (19) is the projection of  $dx$  onto the model plane. In Fig. 1, the projected vector  $P\bar{Q}$  is denoted as  $(du)$ . In terms of the given quantities, the projected vector in covariant components is

$$(du)_\beta = A_\beta^s g_{sr} dx^r = a_\beta^i + b_\beta^j + \dots, \quad (22')$$

which is a tensor equation at  $P$ . The contravariant version of (22') is obtained upon contracting the latter with  $a^{\alpha\beta}$ :

$$(du)^\alpha = a^{\alpha\beta} A_\beta^s g_{sr} dx^r. \quad (22'')$$

This equation gives the result identical to (22). It expresses the L.S. solution of the coordinate differences in the linearized model, as well as the contravariant components of the projection of the vector  $dx$  onto the model plane. Since the L.S. solution of the linearized model is assumed to be close to the final L.S. solution of the nonlinear model, the tensor  $(du)^\alpha$  in (22'') offers a good first approximation to the coordinate differences  $du^\alpha$  in the nonlinear model.

In returning to the covariant components of the projected vector, we rewrite the tensor equation (22') as

$$(du)_\beta = A_\beta^s g_{sr} dx^r. \quad (23a)$$

A similar tensor equation can be formed at the variable point  $\bar{P}$ , namely

$$(d\bar{u})_\beta = \bar{A}_\beta^s g_{sr} d\bar{x}^r. \quad (23b)$$

In the special case where  $d\bar{x}$  is orthogonal to the model surface, the projected vector  $(d\bar{u})$  is zero. The barred tensors on the right-hand side of (23b) can be derived from their unbarred counterparts:

$$\bar{A}_\beta^s = A_\beta^s + dA_\beta^s + \dots, \quad (24a)$$

$$dA_\beta^s = \Omega_{\beta\gamma}^s du^\gamma, \quad (24b)$$

and

$$d\bar{x}^r = dx^r - \Delta x^r, \quad (25a)$$

$$\Delta x^r = dx'^r + d'x'^r + \dots, \quad (25b)$$

how far we can carry the general development without applying a specific criterion for the vector  $\bar{dx}$ .

If the solution of the problem at hand should conform to the L.S. criterion,  $\bar{dx}$  must be orthogonal to the model surface as stated earlier. The variable point  $\bar{P}$  must then coincide with the point denoted  $\bar{P}$  in Fig. 1. Accordingly, the projection of  $\bar{dx}$  on the model surface at  $\bar{P}$  is zero, and thus  $(\bar{du})_\beta$  on the left-hand sides of (23b), (26), and (27) is replaced by zero. At this stage, (27) is contracted with  $a^{\alpha\beta}$ , yielding

$$du^\alpha = (du^\alpha) + a^{\alpha\beta} dA_\beta^s g_{sr} \bar{dx}^r - p_r^\alpha d'x'^r + \dots, \quad (28)$$

where

$$p_r^\alpha = a^{\alpha\beta} A_\beta^s g_{sr}, \quad (29a)$$

which is a tensor at  $P$  that may be called the projection operator (onto the model plane). In terms of this tensor, (22") can be written as

$$(du^\alpha) = p_r^\alpha dx^r, \quad (29b)$$

where  $(du)$  is the projection of  $dx$  onto the model plane. Of the other quantities in (28),  $dA_\beta^s$  is given by (24b),  $\bar{dx}^r$  is given by (25a-d),  $d'x'^r$  is given by (25d) alone, and  $a^{\alpha\beta}$ , the "inverse" of  $a_{\beta\alpha}$ , follows from (20a,b). On the other hand, the tensor  $dx^r$  (corresponding to the observations via equation 5), the tensor  $g^{rs}$  and thus its "inverse"  $g_{sr}$  (corresponding respectively to the variance-covariance and the weight matrices of the observations), the design tensor  $A_\alpha^r$  (corresponding to the design matrix), and the sets of partial derivatives  $\Omega_{\alpha\beta}^r, \dots$ , are known a priori.

All of the tensors in (28) except  $(du^\alpha)$  contain the desired coordinate differences  $du^\alpha$  between the point  $\bar{P}$  representing the L.S. solution and the fixed point  $P$ . And as the preceding paragraph has demonstrated, with the exception of the coordinate differences  $du^\alpha$  themselves, all of the quantities in (28) are known, including the tensor  $(du^\alpha)$  obtained as the solution to the linearized model. The relation (28) thus represents a nonlinear equations in  $u$  unknowns. Since the solution  $(du^\alpha)$  is a good initial approximation to the desired values  $du^\alpha$ , all of the sets  $du^\alpha$  on the right-hand side could be replaced by  $(du^\alpha)$  in what can be termed the first geometrical iteration. Subsequently, the improved set  $du^\alpha$  computed as the left-hand side of (28) can be substituted

straightforward fashion as indicated by the dots. However, this is not foreseen as being necessary or desirable in practice, where the coordinates  $u_0^\alpha$  are usually close to their final values and the terms containing higher than second powers of  $du^\alpha$  are completely negligible. The terms developed explicitly in this study already go one order of refinement beyond the usual linearization.

In transcribing the complete iterative procedure in matrix notations consistent with the above specifications, we begin by grouping the observations in the (column) vector  $x$ , and grouping the observables associated with the coordinates  $u_0^\alpha$  in the (column) vector  $x_0$ . This leads to the formation of the vector  $dx$  according to (5), namely

$$dx = x - x_0, \quad (30a)$$

where each vector has the dimensions  $(n \times 1)$ . The variance-covariance matrix of observations is  $g$ , and their weight matrix is

$$g^* = g^{-1}, \quad (30b)$$

where either matrix has the dimensions  $(n \times n)$ . The design matrix  $A$  of dimensions  $(n \times u)$  corresponds to  $A_\alpha^r$  in (14a):

$$A \equiv [\partial x^r / \partial u^\alpha], \quad (30c)$$

where the brackets indicate grouping of elements, the contravariant indices identifying the rows and the covariant indices identifying the columns. The partial derivatives of the elements in  $A$  are grouped in a three-dimensional array  $\{\partial A / \partial u\}$ , such that

$$\{\partial A / \partial u\} = [\partial A / \partial u^1], [\partial A / \partial u^2], \dots, [\partial A / \partial u^u], \quad (30d)$$

where each pair of brackets defines a matrix of dimensions  $(n \times u)$ , and where these  $u$  matrices are separated from each other by commas. The matrices in (30c,d) are evaluated with the set  $u^\alpha = u_0^\alpha$ . All of the quantities in (30a-d) are considered to be known a priori.

The starting values for the final L.S. solution  $du$  are obtained from the linearized model associated with the above set  $u^\alpha = u_0^\alpha$ . In particular, one computes successively:

With all the quantities for the right-hand side of (28) available and expressed in matrix notations, this equation is transcribed as

$$du = (du) + adA^T g^* d\bar{x} - Pd'x' + \dots \quad (33)$$

The new  $du$  above is used for the right-hand side in the next iteration, which begins with the computation of the updated values in (32a-e). In the first geometrical iteration, the "new"  $du$  is  $(du)$ . When the elements of  $du$  no longer change, or change by negligible amounts, the iterative process is terminated.

## 2.4 Complete Resolution

Upon completion of the iterative process, the final coordinate values describing the desired L.S. point  $\bar{P}$  are

$$u^\alpha(\bar{P}) = u_0^\alpha + du^\alpha, \quad (34)$$

which follows from (8). The geometrical iterations also yield, as a by-product, the values of  $d\bar{x}^r$  corresponding to the minus residuals:

$$d\bar{x}^r = dx^r - \Delta x^r, \quad (35a)$$

where

$$\Delta x^r = dx'^r + d'x'^r + \dots \quad (35b)$$

Equations (35a,b) as well as the expressions for  $dx'^r$  and  $d'x'^r$  have been presented in (25a d). The position vector  $\bar{x}^r$  of the L.S. point  $\bar{P}$  follows from (9) and (11) as

$$\bar{x}^r = x_0^r + \Delta x^r = x^r - d\bar{x}^r. \quad (36)$$

The values of  $\bar{x}^r$  correspond to the adjusted observations.

In addition to the parametric solution, the L.S. adjustment process should also yield the variance-covariance matrices of the adjusted parameters and observations, and possibly of linear functions of the adjusted parameters. To this we shall add the weight matrices of the adjusted parameters and observations. Clearly, all these matrices could be computed by forming the matrix of normal equations corresponding to the point  $\bar{P}$  and inverting it. However, in a consistent effort to avoid any inversion beyond the solution of



In expanding  $a^{\alpha\beta}$  in the Taylor series, we take advantage of the result just derived and of the basic equation (20b), rewritten as

$$a^{\alpha\tau} a_{\tau\epsilon} = \delta_{\epsilon}^{\alpha}.$$

Upon differentiating the latter with respect to  $u^{\gamma}$ , the right-hand side becomes zero. The contraction of the resulting equation with  $a^{\epsilon\beta}$  produces

$$\partial a^{\alpha\beta} / \partial u^{\gamma} = -a^{\alpha\mu} (\partial a_{\mu\epsilon} / \partial u^{\gamma}) a^{\epsilon\beta},$$

and thus

$$da^{\alpha\beta} = (\partial a^{\alpha\beta} / \partial u^{\gamma}) du^{\gamma} = -a^{\alpha\mu} da_{\mu\epsilon} a^{\epsilon\beta},$$

which is symmetric in the indices  $\alpha$  and  $\beta$ , as is confirmed upon using the symmetry in  $da_{\mu\epsilon}$ . If we differentiate  $\partial a^{\alpha\beta} / \partial u^{\gamma}$  with respect to  $u^{\delta}$ , and contract the new equation with  $du^{\gamma} du^{\delta}$ , we obtain

$$\begin{aligned} d'a^{\alpha\beta} &= (1/2)(\partial^2 a^{\alpha\beta} / \partial u^{\gamma} \partial u^{\delta}) du^{\gamma} du^{\delta} \\ &= -a^{\alpha\mu} d'a_{\mu\epsilon} a^{\epsilon\beta} - (1/2)(S^{\alpha\beta} + S^{\beta\alpha}), \end{aligned}$$

where

$$S^{\alpha\beta} = a^{\alpha\mu} da_{\mu\epsilon} da^{\epsilon\beta} = S^{\beta\alpha}.$$

The second equality above is the consequence of the formula giving  $da^{\alpha\beta}$ , and of the symmetry in  $a^{\alpha\mu}$  as well as in  $da_{\mu\epsilon}$ .

In collecting the results, we write the final outcome as

$$\bar{a}^{\alpha\beta} = a^{\alpha\beta} + \Delta a^{\alpha\beta}, \quad (38a)$$

$$\Delta a^{\alpha\beta} = da^{\alpha\beta} + d'a^{\alpha\beta} + \dots, \quad (38b)$$

$$da^{\alpha\beta} = -a^{\alpha\mu} da_{\mu\epsilon} a^{\epsilon\beta}, \quad (38c)$$

$$d'a^{\alpha\beta} = -a^{\alpha\mu} d'a_{\mu\epsilon} a^{\epsilon\beta} - S^{\alpha\beta}, \quad (38d)$$

$$S^{\alpha\beta} = a^{\alpha\mu} da_{\mu\epsilon} da^{\epsilon\beta}, \quad (38e)$$

all of which are symmetric in  $\alpha$  and  $\beta$ . Upon considering (37b), equations (38b-e) could be replaced by a more concise expression:

$$\bar{g}_{sr}'' = \bar{g}_{sr} - \bar{g}_{sr}' = g_{sr} - \bar{g}_{sr}' , \quad (43)$$

where the second equality is again due to the constant  $g_{sr}$ , and  $\bar{g}_{sr}'$  is given above by (42). As a matter of interest, one can form a quantity similar to  $\bar{A}_{\alpha}^r$  in (39), and associate it with linear, or linearized, functions of the adjusted parameters. In this case, all the quantities in (39) except  $\bar{a}^{\alpha\beta}$  would be attributed the symbol  $\sim$ . The resulting tensor at  $\bar{P}$ , called the necessary associated metric tensor in the functional space, would represent the variance-covariance matrix of these functions. However, the weight matrix of such functions could not be expressed in analogy to (42) because the metric tensor in the functional space is unknown.

The key formulas derived in this section will now be transcribed in matrix notations in accordance with the convention introduced earlier. In grouping the  $u$  adjusted parameters in the (column) vector  $\bar{u}$ , one can transcribe (34) as

$$\bar{u} = u_0 + du . \quad (44)$$

The minus residual vector follows from (35a,b) as

$$d\bar{x} = dx - \Delta x , \quad (45a)$$

where

$$\Delta x = dx' + d'x' + \dots \quad (45b)$$

Equations (45a,b) as well as the expressions for  $dx'$  and  $d'x'$  have already appeared in (32b-e). With regard to the adjusted observations, (36) gives

$$\bar{x} = x_0 + \Delta x = x - d\bar{x} . \quad (46)$$

The weight matrix of the adjusted parameters,  $\bar{a}^*$ , follows from (37a-e):

$$\bar{a}^* = a^* + \Delta a^* , \quad (47a)$$

$$\Delta a^* = da^* + d'a^* , \quad (47b)$$

$$da^* = N + N^T , \quad (47c)$$

$$N = A^T g^* dA , \quad (47d)$$

$$d'a^* = dA^T g^* dA . \quad (47e)$$

Finally, the weight matrix of the adjusted observations is transcribed from (42) as

$$\bar{g}^{*'} = g^* \bar{g}' g^* , \quad (52)$$

where  $\bar{g}'$  is given by (49). For the weight matrix of the residuals, from (43) we have

$$\bar{g}^{*''} = g^* - \bar{g}^{*'} , \quad (53)$$

with  $\bar{g}^{*'}$  given above by (52). The weight matrix of functions of the adjusted parameters cannot be computed in analogy to (52) because the weight matrix in the "functional space" is unknown.

As a matter of interest, we present a formula serving for the verification of the final L.S. outcome. This relation has been encountered in tensor form as a by-product of the geometrical development in Section 2.3. In particular, equation (23b) and the statement immediately following it have established that if the vector  $d\bar{x}$  is orthogonal to the model surface, the vector  $(d\bar{u})$  is zero. But this occurs precisely when the variable point  $\bar{P}$  mentioned prior to (23b) becomes the desired L.S. point  $\bar{P}$ . Accordingly, upon the convergence of the L.S. process, it must hold true that

$$\bar{A}^T g^* d\bar{x} = 0 ,$$

which is the matrix form of (23b) with  $(d\bar{u}_\beta)=0$ .

du implicated in the second, third, and further terms on the right-hand side is (du). The quantities computed with this vector will be symbolized here by parentheses as well. Thus, we write

$$du_{(1)} = (du) + a(dA^T)g^*(d\bar{x}) - aA^Tg^*(d'x') + \dots, \quad (57)$$

where

$$(dA) = \{\partial A / \partial u\}(du), \quad (58a)$$

$$(d\bar{x}) = dx - (dx') - (d'x') - \dots, \quad (58b)$$

$$(dx') = A(du), \quad (58c)$$

$$(d'x') = (dA)(du)/2. \quad (58d)$$

Equation (58a) stems from (32a), and equations (58b-d) stem from (32b-e), with the parentheses introduced in accordance with the above convention.

We now turn our attention to the first adjustment iteration and compare the result with that produced by the first geometrical iteration. In analogy to (54), the outcome of the first adjustment iteration is represented by

$$(du)_{(1)} = a_{(0)}A_{(0)}^Tg^*dx_{(0)}, \quad (59)$$

where the subscript (0) on the right-hand side indicates that the pertinent quantities are associated with the point  $P_0$  generated in the zero-th iteration. To establish a theoretical link with the geometrical approach, we express these quantities in terms of their counterparts at P via the Taylor series expansion, using the coordinate differences (du) relating  $P_0$  to P:

$$a_{(0)} = a + (da) + \dots, \quad (60a)$$

$$A_{(0)} = A + (dA) + \dots, \quad (60b)$$

$$dx_{(0)} = dx - (dx') - (d'x') - \dots = (d\bar{x}). \quad (60c)$$

The parentheses serve the same purpose as in the previous paragraph. The first equality in (60c) is essentially (25a,b), where the variable point P is taken as the current point  $P_0$ . The second equality in (60c) stems from (58b).

### 3.2 Second and Further Iterations

We now return to the formula (56) and apply it to the second geometrical iteration. In this case,  $du$  on the left-hand side is written as  $du_{(2)}$ , while  $du$  on the right-hand side is  $du_{(1)}$ . However, due to (64),  $du$  on the right-hand side is replaced by

$$(du) + (du)_{(1)} = [du] . \quad (66)$$

Upon symbolizing the quantities computed with  $[du]$  by brackets, in analogy to (57) and (58a-d) we have

$$du_{(2)} = (du) + a[dA^T]g^*[d\bar{x}] - aA^Tg^*[d'x'] + \dots , \quad (67)$$

where

$$[dA] = \{\partial A / \partial u\}[du] , \quad (68a)$$

$$[d\bar{x}] = dx - [dx'] - [d'x'] - \dots , \quad (68b)$$

$$[dx'] = A[du] , \quad (68c)$$

$$[d'x'] = [dA][du]/2 . \quad (68d)$$

With regard to the second adjustment iteration, in analogy to (59) we write

$$(du)_{(2)} = a_{(1)}A_{(1)}^Tg^*dx_{(1)} , \quad (69)$$

where the subscript (1) indicates the quantities associated with the point  $P_1$  generated in the first iteration. Again, these quantities are expressed via the Taylor series expansion at  $P$ , this time upon using the coordinate differences  $(du) + (du)_{(1)}$  relating  $P_1$  to  $P$  according to (65), i.e., using  $[du]$  from (66):

$$a_{(1)} = a + [da] + \dots , \quad (70a)$$

$$A_{(1)} = A + [dA] + \dots , \quad (70b)$$

$$dx_{(1)} = dx - [dx'] - [d'x'] - \dots = [d\bar{x}] . \quad (70c)$$

The first equality in (70c) again corresponds to (25a,b), this time with the variable point  $\bar{P}$  taken as  $P_1$ , while the second equality stems from (68b).

The second adjustment iteration generates the point  $P_2$ , whose model-surface coordinates are

$$u(P_2) = u(P_1) + (du)_{(2)} ,$$

or, upon considering (65),

$$u(P_2) = u(P) + (du) + (du)_{(1)} + (du)_{(2)} . \quad (75)$$

On the other hand, the second geometrical iteration gives rise to the point whose model-surface coordinates are  $u(P) + du_{(2)}$ . However, due to (74') we have

$$u(P) + du_{(2)} = u(P) + (du) + (du)_{(1)} + (du)_{(2)} ,$$

revealing that the second geometrical and adjustment iterations lead essentially to the same point  $P_2$ .

In proceeding in the same fashion, we would find for the third iteration:

$$[I + \{da\}a^*][(du)_{(1)} + (du)_{(2)}] + (du)_{(3)} = du_{(3)} + (du) . \quad (76)$$

Here  $\{da\}$  is a function of  $\{du\}$ , such that

$$(du) + (du)_{(1)} + (du)_{(2)} = \{du\} ,$$

which represents the coordinate differences between  $P_2$  and  $P$ . Again, if the magnitude of  $(du)_{(3)}$  drops significantly vis-a-vis  $(du)_{(2)}$ , the iterative process has already converged. Otherwise the terms with  $\{da\}$  on the left-hand side of (76) can be disregarded, yielding

$$(du)_{(1)} + (du)_{(2)} + (du)_{(3)} = du_{(3)} + (du) . \quad (76')$$

The third adjustment iteration generates the point  $P_3$ , whose model surface coordinates are

$$u(P_3) = u(P) + (du) + (du)_{(1)} + (du)_{(2)} + (du)_{(3)} . \quad (77)$$

On the other hand, the third geometrical iteration gives rise to the point with the model surface coordinates

$$u(P) + du_{(3)} = u(P) + (du) + (du)_{(1)} + (du)_{(2)} + (du)_{(3)} ,$$

where (76') has been taken into account. But this means that the third geometrical and adjustment iterations also lead essentially to the same point.

#### 4. CONCLUSIONS

The isomorphism between a nonlinear adjustment model with  $n$  observations and  $u$  parameters on one hand, and a  $u$ -dimensional model surface embedded in an  $n$ -dimensional (flat) observational space on the other, has been described in the Introduction. To avoid unnecessary repetitions, it is only mentioned that in the geometrical context, the observations correspond to the observational space coordinates of the point called  $Q$  and the parameters correspond to the model surface coordinates of the point called  $\bar{P}$ . The latter can be perceived as a result of mapping of  $Q$  onto the model surface according to a certain criterion.

The isomorphic geometrical setup has been developed in Chapter 2 by means of tensor structure and notations. Crucial to this development are three geometrical notions:

- 1) The fundamental tensors  $g^{rs}$  and  $g_{sr}$  are constant throughout the observational space. In the adjustment context, these tensors correspond respectively to the variance-covariance and the weight matrices of observations.

- 2) The quantities such as  $du^\alpha$  and  $dx^r$  are regarded not only as sets of model-surface and observational-space coordinate differences, respectively, but also as contravariant components of small vectors (geometrical objects). In the adjustment context, the components  $du^\alpha$  correspond to the adjusted parameters referring to the "initial point"  $P$ , where the model is expanded in the Taylor series, and the components  $dx^r$  correspond to the observations referring to  $P$ .

- 3) The simplest and the most objective mapping of  $Q$  onto the model surface is the one where the distance from  $\bar{P}$  to  $Q$  is a minimum, in which case the small vector  $PQ$ , denoted  $\bar{dx}$ , is orthogonal to the model surface. Since, in the adjustment context, the length square of  $\bar{dx}$  corresponds to the standard quadratic form  $V^T P V$ , the orthogonality condition characterizes the least squares (L.S.) criterion, and, as such, is adhered to throughout this study.

The primary objective in the geometrical resolution of a nonlinear model is the determination of  $du^\alpha$ , the coordinate differences between the L.S. point  $\bar{P}$  and the initial point  $P$ . An essential stepping stone in this task, accomplished in Chapter 2, is the determination of  $(du^\alpha)$  which represents the L.S. solution of the corresponding linearized model. In the latter, the role of the model surface is substituted for by the model plane, i.e., a  $u$  dimensional plane

such vectors are replaced by the new result, etc. However, the term  $(du)$  on the right-hand side is fixed throughout. When the results no longer change, or change by insignificant amounts, the convergence has been achieved.

In using the above algorithm, one can compute the value of the L.S. quadratic form in every iteration, corresponding to the length square of the vector  $\bar{P}Q$ . The latter is given in matrix notations as  $dx^T g dx$ . This computation is virtually effortless because the product  $g dx$  must be evaluated in any event as a part of the iterative process. The knowledge of the L.S. quadratic form at every step is especially useful if the possibility exists that the solution could converge to a maximum rather than to a minimum. Since the current algorithm is based on the orthogonality condition, it is unable to distinguish between the two extremes in the case where both occur within a small neighborhood of  $P$ . The computation of the L.S. quadratic form can quickly discern such a difficulty, which can then be rectified upon choosing a different initial point  $P$ .

A comparison between the new "geometrical iterations" and the standard "adjustment iterations" has been carried out in Chapter 3 and illustrated in the Appendix. It has been shown that the two processes converge at almost exactly the same pace, generating nearly identical model-surface points at every step beyond the common zero-th iteration. This similarity in outcome occurs in spite of a significant operational disparity. Whereas each adjustment iteration involves a numerical reformulation of the linearized model, including the inversion of the matrix of normal equations, the geometrical iterations keep the quantities which are independent of the solution fixed. The benefit of the geometrical approach thus becomes clearer. Unlike the standard procedure, this approach does not benefit from a fresh, albeit expensive, start at every step, yet it produces the same result in the same number of iterations.



The actual observations of  $x^r(u^\alpha)$ , grouped in the column vector  $x$ , are stipulated to be

$$x = \begin{bmatrix} 1.00 \\ 0.22 \\ 8.32 \end{bmatrix} . \quad (A.4)$$

The initial point  $P$  is described by the column vector  $u_0$ . The latter contains the model-surface coordinates corresponding to an initial set of parameters:

$$u_0 = \begin{bmatrix} 1.19 \\ 1.29 \end{bmatrix} . \quad (A.5)$$

In the observational-space coordinates,  $P$  is described by the column vector  $x_0$  obtained through (A.1a-c) with the values of  $u^1$  and  $u^2$  from (A.5):

$$x_0 = \begin{bmatrix} 1.19 \\ 1.29 \\ 8.0925 \end{bmatrix} . \quad (A.6)$$

The basic adjustment setup consists of  $dx$ ,  $g^*$ ,  $A$ , and  $\{\partial A/\partial u\}$ , defined in (30a-d). The values in  $dx$  follow from (A.4) and (A.6), namely

$$dx = x - x_0 = \begin{bmatrix} -0.19 \\ -1.07 \\ 0.2275 \end{bmatrix} . \quad (A.7)$$

According to an earlier statement,  $g$  and  $g^*$  are given by

$$g = g^* = I . \quad (A.8)$$

Upon substituting the values from (A.5) into (A.2), the design matrix associated with  $P$  becomes

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1.98 & 10.72 \end{bmatrix} . \quad (A.9)$$

The array  $\{\partial A/\partial u\}$  is constant, and has already been presented in (A.3a-c).

If the model-surface coordinates (A.11) are utilized also in (A.2), the design matrix becomes

$$A(\text{new}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1.9966618 & 10.8028834 \end{bmatrix}. \quad (\text{A.13})$$

In (A.12) and (A.13), the symbol (new) is used in lieu of (0) as it appeared e.g. in equation (59).

In paralleling (A.10a-c), one computes the pertinent matrices in the first adjustment iteration:

$$a^*(\text{new}) = \begin{bmatrix} 4.9866584 & 21.5697048 \\ 21.5697048 & 117.7022895 \end{bmatrix},$$

$$a(\text{new}) = \begin{bmatrix} 0.9672389 & -0.1772528 \\ -0.1772528 & 0.0409787 \end{bmatrix},$$

$$P(\text{new}) = \begin{bmatrix} 0.9672389 & -0.1772528 & 0.0164079 \\ -0.1772528 & 0.0409787 & 0.0887746 \end{bmatrix}.$$

The solution vector in this iteration is denoted  $(du)_{(1)}$  in agreement with the notation in Chapter 3. Similar to (A.10d), but with  $P(\text{new})$  and  $dx(\text{new})$  replacing  $P$  and  $dx$ , this vector is computed as

$$(du)_{(1)} = \begin{bmatrix} 0.0001354 \\ 0.0000006 \end{bmatrix}. \quad (\text{A.14})$$

The updated point  $P$  in the first adjustment iteration is denoted  $P_1$ , again in agreement with Chapter 3. Its model-surface coordinates are obtained by adding (A.14) to (A.11):

$$u(P_1) = \begin{bmatrix} 1.1996260 \\ 1.3000711 \end{bmatrix}. \quad (\text{A.15})$$

The second adjustment iteration proceeds in analogy to the preceding two paragraphs, except that it is based on (A.15) instead of (A.11). The final results paralleling (A.14) and (A.15) are

where the array  $\{\partial A/\partial u\}$ , the matrix  $A$ , and the vector  $dx$ , presented respectively in (A.3a-c), (A.9), and (A.7), are fixed. Since the current adjustment model is quadratic, the dots in (A.20) and (A.21d) can be discarded.

In the first geometrical iteration,  $du$  in (A.21a-c) is replaced by  $(du)$  from (A.10d). The indicated operations result in

$$dA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.0166618 & 0.0828834 \end{bmatrix};$$

$$dx' = \begin{bmatrix} 0.0094906 \\ 0.0100705 \\ 0.1267471 \end{bmatrix}, \quad d'x' = \begin{bmatrix} 0 \\ 0 \\ 0.0004964 \end{bmatrix}, \quad d\bar{x} = \begin{bmatrix} -0.1994906 \\ -1.0800705 \\ 0.1002565 \end{bmatrix}.$$

The utilization of the new quantities  $dA$ ,  $d'x'$ , and  $d\bar{x}$  in (A.20) yields the solution in the first geometrical iteration:

$$du_{(1)} = \begin{bmatrix} 0.0096265 \\ 0.0100714 \end{bmatrix}. \quad (A.22)$$

This result gives rise to a point described by the model surface coordinates

$$u_o + du_{(1)} = \begin{bmatrix} 1.1996265 \\ 1.3000714 \end{bmatrix}, \quad (A.23)$$

where  $u_o$  has been listed in (A.5). Upon consulting (A.15), the new point is seen to be nearly identical to  $P_1$  generated in the first adjustment iteration.

The second geometrical iteration follows the procedure just described, except that  $du$  in (A.21a-c) is now replaced by  $du_{(1)}$ . The solution is

$$du_{(2)} = \begin{bmatrix} 0.0097142 \\ 0.0100544 \end{bmatrix}, \quad (A.24)$$

which, when added to  $u_o$  in (A.5), gives rise to another point, whose model surface coordinates are

$$u_o + du_{(2)} = \begin{bmatrix} 1.1997142 \\ 1.3000544 \end{bmatrix}. \quad (A.25)$$

#### A.4 Adjustment Results and Verifications

In using (A.28) in (A.21a-e), one obtains

$$dA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.02 & 0.08 \end{bmatrix}; \quad (A.30)$$

$$dx' = \begin{bmatrix} 0.01 \\ 0.01 \\ 0.127 \end{bmatrix}, \quad d'x' = \begin{bmatrix} 0 \\ 0 \\ 0.0005 \end{bmatrix}, \quad d\bar{x} = \begin{bmatrix} -0.20 \\ -1.08 \\ 0.10 \end{bmatrix}.$$

The column vector  $d\bar{x}$  represents the minus residuals. The final L.S. quadratic form is

$$d\bar{x}^T g^* d\bar{x} = 1.2164.$$

Although not listed, this quadratic form has been computed throughout the iterative process, and has been noted to converge to its minimum value presented above. The adjusted parameters are given by (A.29) as

$$\bar{u} = \begin{bmatrix} 1.20 \\ 1.30 \end{bmatrix}, \quad (A.31)$$

while the adjusted observations follow from (46) as

$$\bar{x} = \begin{bmatrix} 1.20 \\ 1.30 \\ 8.22 \end{bmatrix}. \quad (A.32)$$

Equation (A.32) is confirmed by the direct evaluation of (A.1a-c) with the values of  $u^1$  and  $u^2$  from (A.31).

In adding  $dA$  from (A.30) to  $A$  given by (A.9), one obtains the design matrix at the L.S. point  $\bar{P}$ :

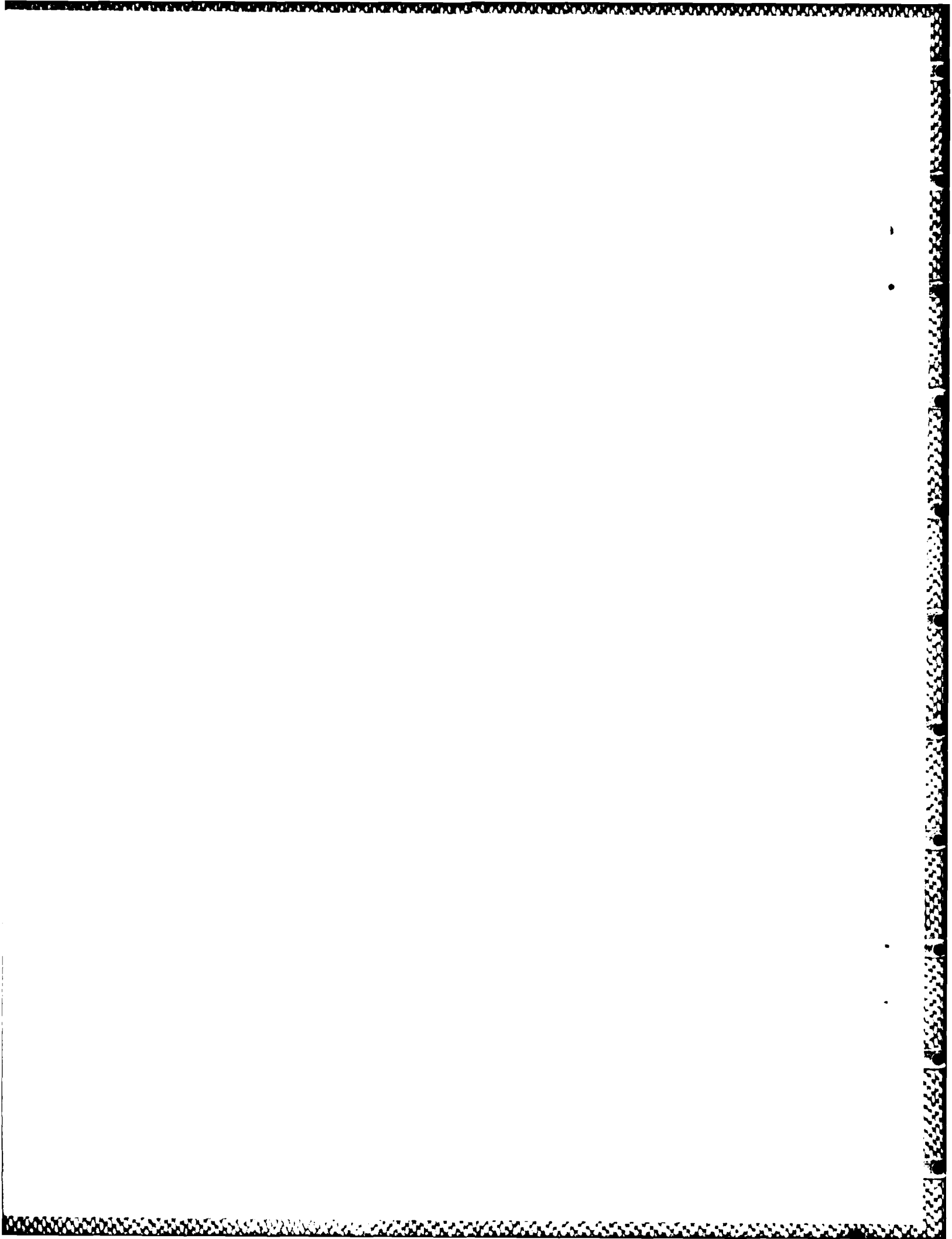
$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 10.8 \end{bmatrix}. \quad (A.33)$$

This outcome is also established directly upon utilizing the values of  $u^1$  and  $u^2$  from (A.31) in (A.2). A useful verification is offered by

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